Limitations on Brandão-Harrow limitation for 4-local Hamiltonians

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Given n qubits, a (k,d)-local Hamiltonian is $H = \frac{1}{m}\sum_{\alpha=1}^{m} h_\alpha$ (with $0 \preceq h_\alpha \preceq 1$) such that each h_α acts non-trivially on $\leq k$ qubits and each qubit participates in $\leq d$ terms. The smallest eigenvalue of H, $E_0(H)$, is its ground energy.

Brandão and Harrow [BH13] showed that given a $(2,d)$-local Hamiltonian, there exists a product state $|\psi\rangle = \otimes_{i=1}^{n} |\psi_i\rangle$ such that

$$\langle\psi| H |\psi\rangle \leq E_0(H) + \mathcal{O}(1/d^{1/3}).$$

This is a powerful bound and explains the mean-field approximation in physics. It limits quantum generalizations to the classical PCP gap amplification [Din07], since the latter increases the degree a lot but keeps locality at 2.

Here, we show that similar result (in terms of dependence on degree, which is the main quantity of interest) is not possible when $k \geq 4$. For this, consider any $(2,2)$-local Hamiltonian G which has ground energy $E_0(G) = 0$, minimum product energy p for a constant $p = \Omega(1)$. An example is the AKLT hamiltonian [AKLT87] in 1D, which has $m = n$. Consider the Hamiltonian $F = G^2$, which is a $(4,n)$-local with $m = n^2$ terms. Note that $E_0(F) = E_0(G) = 0$. We argue that for any product state $|\psi\rangle = \otimes_{i=1}^{n} |\psi_i\rangle$, $\langle\psi| F |\psi\rangle = p^2 - o_n(1) = \Omega(1)$, which shows that the product state energy does not approach 0, despite the degree increasing with n.

To lower bound the product state energy, note that the energy of $|\psi\rangle$ with respect to G concentrates about the average [Ans16, Kuw16]. More precisely, consider the eigen-decomposition of $G = \sum_r \lambda_r \Pi_r$ and let $\mu = \langle\psi| G |\psi\rangle$ (note that $\mu \geq p$). There is a constant c such that

$$\langle\psi| \left(\sum_{r:|\lambda_r - \mu| \geq cn^{-0.1}} \Pi_r \right) |\psi\rangle \leq e^{-n^{0.8}}.$$

Thus,

$$\langle\psi| G^2 |\psi\rangle = \sum_r \lambda_r^2 \langle\psi| \Pi_r |\psi\rangle \geq (\mu - cn^{-0.1})^2 (1 - e^{-n^{0.8}}) = \mu^2 - o_n(1) \geq p^2 - o_n(1).$$

This completes the argument.

References

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